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Fractional Hall conductance in a set of quantum many-vortex Hamiltonians

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Abstract. A quantum Hamiltonian describing a set of interacting vortices is studied. A non-relativistic approximation for anomalous quantum electrodynamics in (2+1) dimensions is provided by this type of Hamiltonian. This model leads to fractional values of Hall conductance once the assumption that the external magnetic field pierces the plane creating the phenomenological vortices is granted. Conditions imposed on the wavefunction lead to a transverse conductivity which is quantised at least in the values $\sigma = (1/N)e^2/h$. N can be restricted to be an odd integer.

1. Introduction

In this paper I show that the mechanism proposed by Friedman *et al* [1] to generate a fractional Hall conductance from anomalous quantum electrodynamics in (2+1) dimensions (AQED) can be extended to other theories, namely the quantum many-vortex Hamiltonian (QMVH) defined here.

The approach is based on the existence of extended carriers which exist whenever the external magnetic field satisfies a relation resembling magnetic flux quantisation. This takes place in the moving frame and by examining the physical quantities at the laboratory a fractional transverse conductance naturally appears. The discussion of this model-independent mechanism is the subject of § 2. In § 3, I show that the QMVH can make use of it. There the vortex concept is introduced. The vortex is intended to be the extended carrier of § 1 and must be interpreted as a quasi-particle describing a charged particle and its surrounding magnetic field. The non-relativistic Schrödinger equation describing the mutual vortex interaction, the QMVH, is constructed. I show that the elimination of fractional statistics, or instead the requirement of single-valuedness, leads to the magnetic flux quantisation assumed in § 1. For fermionic vortices the ground state of the QMVH can have the interesting feature that antisymmetry holds without the presence of nodes in the wavefunction. In § 4, AQED in a particular non-relativistic limit is shown to become a QMVH. In this sense the present work broadens the approach of Friedman *et al* [1] to obtain fractional Hall conductance. In the conclusion, the assumptions required to interpret the vortex flux quantisation in a QMVH as a fractional Hall conductance are listed. The possibility of applying this theory to the two-dimensional electron gas in a magnetic field is discussed and the main obstacles are outlined.

Theories similar to the QMVH have been discussed before†. Here I show how they can lead to fractional Hall conductance. The present QMVH mechanism captures

† See for instance [2].

elements of both Laughlin's original wavefunction proposal [3] and the more recent suggestions of a possible connection between AQED and the fractional quantum Hall effect [1, 4].

2. Fractional transverse Hall conductance

In this section, an external magnetic flux quantisation type of relation is shown to lead to the discretisation of the transverse Hall conductance in fractional integer values when properly interpreted.

Let axis 1 denote the direction where current is flowing and axis 2 the transverse direction orthogonal to it (see figure 1).

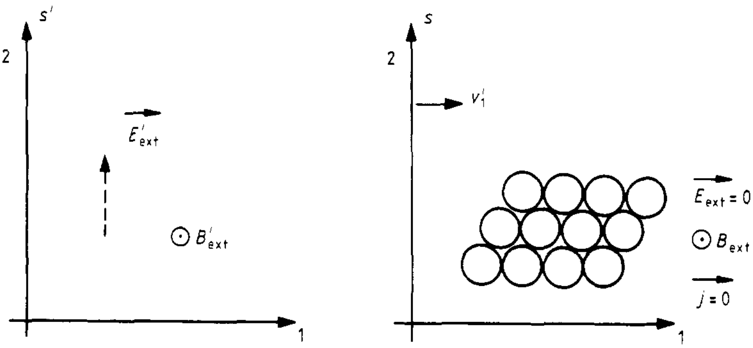

Figure 1. The extended carriers have no macroscopic motion in the moving frame. They are pictorially represented by the circles in a tight packing arrangement.

The density of charge per area n is always constant. I assume that this charge flow is made of *extended carriers*, each one occupying a characteristic area $S = 1/n$. For a system with total area A and K of such extended carriers it follows that $S = A/K$ too. Therefore such extended carriers are in a tight packing arrangement (the dynamical reason for this is not formulated in this paper). The charge of these extended carriers is taken to be any integer multiple of the fundamental charge.

There are two frames of interest here, the *laboratory* and *moving frames*. Attach a prime mark ' ' ' to identify all the quantities measured in the laboratory frame, where all the measurements are performed. There an external magnetic field is applied and a transverse electric field is established once the equilibrium transport sets in. The extended carriers move in the longitudinal direction with a uniform *macroscopic* constant current density j_1'. The moving frame is defined such that there is no transverse electric field. Assume that in this frame all *macroscopic* currents are totally negligible ($j_1 = j_2 = 0$).

The main hypothesis of this section is that in the moving frame the following magnetic flux relation ought to be satisfied:

$$B_{ext} S = Nhc/e \tag{1}$$

N being any positive integer. As described before, S is the intrinsic area of the extended carrier. Notice that the external magnetic field can be varied continuously. Thus the above relation means that such extended carriers can exist only when the external magnetic field fulfils it for a particular integer N. This expression can be conveniently written as $n/B_{ext} = (1/N)e/hc$. Now it is just a matter of performing a boost transformation to the laboratory frame to show that

$$\sigma = (1/N)e^2/h \tag{2}$$

follows from (1). The condition that the moving frame feels no external electric field in the transverse direction, immediately tells us the velocity with which it moves with respect to the laboratory†: $v'_1/c = E'_{2ext}/B'_{ext}$. It also follows from the same Lorentz boost that $B'_{ext} = \gamma B_{ext}$. Therefore as seen in the laboratory frame, there is a current density moving along direction 1: $j'_1 = \rho' v'_1$, $\rho' = \gamma\rho$ so that $j'_1 = c\sigma_{12}E'_{2ext}$ where $\sigma_{12} = \rho'/B'_{ext}$ ($\rho = en$). One can readily check that this quantity is an invariant, $\rho'/B'_{ext} = \rho/B_{ext}$.

If one allows the existence of small fluctuations on the carrier's intrinsic area S, a natural plateau width results when the external magnetic field is varied. Beyond the boundaries of this plateau, the extended carriers cannot live due to the flux quantisation requirement.

3. Quantum many-vortex Hamiltonian

This section's goal is to show that the intrinsic magnetic flux created by a vortex of the QMVH must be quantised. Thus the QMVH leads to relation (1) provided that some further considerations are added to its interpretation. The values of the integer N can be restricted to be odd as we shall see. For fermionic vortices the ground-state wavefunction can have the remarkable property of being antisymmetric and at the same time having no zeros at all! The wavefunction is taken to be totally symmetric in the spin degrees of freedom. This means total polarisation in the presence of an external magnetic field if spin is present at all.

Assume the existence of a preferred plane containing the vortices. In this plane, the vortex is taken to be a source of an electric field and of an intrinsic magnetic field orthogonal to it. For the fields produced by this vortex, take $V(r)$ to be the electrostatic potential. The intrinsic magnetic field $B(r)$ must be localised, r being the distance in this plane to the centre of the vortex. Thus the divergence of this magnetic field remains null in this two-dimensional geometry. The vortices shall be interpreted as the extended carriers described in the last section. To have such an interpretation, one must regard the vortex intrinsic magnetic field as being a localised external magnetic field. So the situation resembles the Abrikosov vortices in type II superconductors.

Now the QMVH is constructed under the following assumptions: the charged vortices are the only existing sources of magnetic field; they interact and can have their trajectories modified but the electromagnetic field of each one of them remains unchanged when seen in their own rest frame. Therefore in this approximation no radiation emitted or absorbed by the vortices is considered. In fact the electromagnetic field is classical and of the action-at-a-distance type. In this sense the situation is analogous to the problem of treating a K electron atom. There the radiation contribution is as a rule totally ignored and one concentrates on the mutual Coulomb interaction among the electrons and nuclei. Thus the QMVH describing the dynamics of the mutual

† The Lorentz transformations for the current vector and the electromagnetic tensor are as follows: $E_1 = E'_1$, $E_2 = \gamma(E'_2 - \beta B'_3)$, $B_3 = \gamma(B'_3 - \beta E'_2)$, $\rho = \gamma(\rho' - \beta j'_1/c)$, $j_1/c = \gamma(j'_1/c - \beta\rho')$, $j_2 = j'_2$, $\beta = v'_1/c$ and $\gamma = (1-\beta^2)^{-1/2}$.

vortex interactions with no external electromagnetic fields present† is

$$H = \sum_i [(\boldsymbol{P}_i - e\boldsymbol{A}_i/c)^2/2m + eV_i/2]. \tag{3}$$

The electromagnetic field felt by vortex i in the presence of the remaining $K-1$ vortices is given by

$$V_i = \sum_{j\neq i} V(\boldsymbol{r}_{ij}) \qquad \boldsymbol{A}_i = \boldsymbol{A}_{0i} + \boldsymbol{A}_{1i} \qquad \boldsymbol{A}_{0i} = \sum_{j\neq i} \boldsymbol{A}_0(\boldsymbol{r}_{ij}) \qquad \boldsymbol{A}_{1i} = \sum_{j\neq i} \boldsymbol{A}_1(\boldsymbol{r}_{ij}) \tag{4}$$

where the functions V, $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ are the fields produced by a single vortex. $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ is the distance between vortices i and j and $\theta_{ij} = \tan^{-1}[(x_{2i} - x_{2j})/(x_{1i} - x_{1j})]$. The magnetic potential of each vortex has two contributions:

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}_0(\boldsymbol{r}) + \boldsymbol{A}_1(\boldsymbol{r}) \qquad \boldsymbol{A}_0 = A_\theta(r)\boldsymbol{\theta} \qquad \boldsymbol{A}_1 = (\Phi_\mathrm{v}/2\pi)\nabla\theta. \tag{5}$$

The first term is derived by looking at the curl of $\boldsymbol{A}[\partial(rA_\theta)/r\,\partial r - \partial A_r/r\,\partial\theta = B(r)]$ together with the appropriate symmetry arguments $(A_\theta(r), A_r = 0)$. The second term guarantees that at infinity the vortex magnetic flux, Φ_v, is obtained. θ is the azimuthal angle with respect to the x_1 axis. In the Schrödinger equation for this assembly of quantum vortices, $H\Psi = \mathrm{i}\hbar\,\partial\Psi/\partial t$, a gauge transformation rotates away the long-ranged magnetic potential $\boldsymbol{A}_1$ and introduces a potentially multivalued wavefunction,

$$\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k; t) = [\exp(\mathrm{i}ef/\hbar c)]\chi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k; t) \qquad f = (\Phi_\mathrm{v}/2\pi)\sum_{i<j}\theta_{ij}. \tag{6}$$

The demand of single-valuedness for Ψ is an outside physical requirement that is imposed upon the theory‡. The wavefunction χ obeys $h_0\chi = \mathrm{i}\hbar\,\partial\chi/\partial t$, $h_0 = \Sigma_i\,[(\boldsymbol{P}_i - e\boldsymbol{A}_{0i}/c)^2/2m + eV_i/2]$, which contains only short-ranged potentials.

The quantisation of the vortex intrinsic magnetic field follows from imposing the wavefunction χ to be single-valued or instead demanding conventional statistics for the vortices. By requiring that χ be single-valued, the energy levels of (3) become independent of the long-ranged pure gauge term $\boldsymbol{A}_1$. Writing the wavefunction Ψ with complex coordinates, $z_j = x_{1j} + \mathrm{i}x_{2j}$, such that $z_i - z_j = r_{ij}\exp\mathrm{i}\theta_{ij}$ and $r_{ij} = |\boldsymbol{r}_{ij}|$, one finds

$$\Psi = \prod_{i<j}(z_i - z_j)^m\psi \qquad \psi = \chi(z_i^*, z_i)\left(\prod_{i<j} r_{ij}^m\right)^{-1} \tag{7}$$

where $m = e\Phi_\mathrm{v}/hc$. Rotating vortex i around vortex j by an angle $2\pi(\theta_{ij} \to \theta_{ij} + 2\pi)$ such that no other vortices are contained inside the loop brings an extra contribution to the wavefunction: $\Psi(2\pi) = [\exp(\mathrm{i}\,2\pi e\Phi_\mathrm{v}/hc)]\Psi(0)$. This wavefunction being single-valued imposes that the phase must be $2\pi N$, with N a non-zero integer, thus leading to the main result of this section, i.e. the quantisation of the vortex intrinsic magnetic flux:

$$\Phi_\mathrm{v} = Nhc/e. \tag{8}$$

† In principle a neutralising background can be introduced but I do not do so because it is irrelevant for the purposes of this paper.

‡ Merzbacher [5] observes (p 243) that the question of Ψ being single-valued '... can only be answered in terms of a model of the physical situation at hand.... If observation calls for the use of a model in which a portion of space is actually and permanently "off limits" to a particle, the possibility of multivalued wavefunctions would have to be examined, but as long as we think of such restrictions merely as limiting cases of high, finite, but "in principle" penetrable barriers, the wavefunction must be taken to be single-valued in the discussion of the Aharonov–Bohm effect.'

The quantisation of the vortex magnetic field can follow from another condition replacing single-valuedness on χ: fractional statistics [6] is excluded from this model whenever Φ_v is quantised according to the values from (8). For simplicity, consider the presence of only two vortices, i and j. Rotate vortex i around vortex j of an angle π, ($\theta_{ij} \to \theta_{ij} + \pi$, $\theta_{ji} \to \theta_{ji} + \pi$) and require that the wavefunction changes at most by one of the phases $+1$ or -1. Apart from an overall translation, this restriction implies either fermionic or bosonic statistics. Conventional statistics must be valid in both gauges, thus being required for Ψ and χ independently. In (7), this operation corresponds to the exchange $z_i \leftrightarrow z_j$ between the two vortices.

If the wavefunction is given by (6), one can further restrict the values of N to be *odd* integers by including the following hypothesis.

(i) The vortices are fermions.

(ii) If the vortices have spin, the wavefunction contribution is symmetric under the exchange of any two of them.

(iii) There is an energy gap separating the ground and the first excited states so that only the properties of the ground state concern us.

Given these elements, the only possible antisymmetric contribution to the wavefunction comes from this phase factor. Exchanging any two vortices on it ($\theta_{ji} = \theta_{ij} + \pi$) gives $\Psi_{i\leftrightarrow j} = (-1)^N \Psi$, thus restricting N to be odd. Another implicit assumption is being made here. The ground state of the Hamiltonian h_0 must be a completely even function under the interchange of any two particles. This is a fair hypothesis considering that odd functions must vanish at crossing points and this costs kinetic energy, thus increasing the value of the total energy. Remarkably, antisymmetry can be satisfied without the presence of nodes in the wavefunction. Those may appear for the sole reason of a very repulsive core in the electrostatic interaction. Thus the fermionic vortex system displays the properties of a bosonic state.

It is possible to enlarge the allowed values of the magnetic flux to $\Phi_v = (N/P)hc/e$ if the vortices can have a charge Pe where P is an integer. However, notice that the discrete values given by (8) still are a sufficient condition (although not necessary!) to keep the wavefunction single-valued.

As a concrete illustration of a vortex, take the magnetic field $B(r) = B_0 \exp[-(r/r_0)^2]$. This particular magnetic field distribution will be called the phenomenological vortex. From it the magnetic flux is directly computed, $\Phi(r) = B_0 \pi r_0^2 \{1 - \exp[-(r/r_0)^2]\}$. The magnetic potential consistent with these expressions is $A_\theta(r) = -(B_0 r_0^2/2)\{\exp[-(r/r_0)^2]\}/r$. Hence the net magnetic vortex flux, $\Phi_v = B_0 \pi r_0^2$, must be taken in quantised values according to (8).

4. Anomalous quantum electrodynamics in (2 + 1) dimensions

In the non-relativistic regime explained in the last section, one verifies that AQED falls into the class of the QMVH. In this sense the AQED vortex proposed by Friedman *et al* [1] is a particular example of a QMVH with specific electrostatic and magnetic potentials which in fact are singular at their origins. To be able to claim that AQED describes matter in the presence of an external magnetic field, one must evoke assumption (ii) of the conclusion. This is because, contrary to Maxwell's equations in (3 + 1) dimensions, the AQED equations do not admit a constant magnetic field as a vacuum solution ($\rho = j^i = 0$). In fact all time-independent vacuum solutions (apart from null electric and magnetic fields) grow exponentially with distance. Therefore the magnetic field B can only be produced by the matter distribution. The AQED model [1] has the fermions as well as their electromagnetic field confined to the plane. In the QMVH approach of this paper, ones sees that it is possible to evade the restriction of a two-dimensional electromagnetic field simply because one has the freedom of choosing the vortex potentials.

It is well known [7] that the AQED interaction is short ranged due to the presence of a parameter μ † that has the dimension of inverse of length in two-dimensional units. For AQED the analogues of Maxwell's equations with sources become: $\partial E^i/\partial x^i + \mu B = \rho$, $\partial E^2/\partial x^1 - \partial E^1/\partial x^2 + \partial B/c\,\partial t = 0$, $\partial B/\partial x^2 - \partial E^1/c\,\partial t + \mu E^2 = j^1/c$ and $-\partial B/\partial x^1 - \partial E^2/c\,\partial t - \mu E^1 = j^2/c$. The field produced by the static point charge ($\rho(\boldsymbol{r}) = e\delta^2(0), j^i = 0$) of this theory is precisely an example of the vortex defined in the last section: $V(r) = (e/2\pi)K_0(\mu r)$ and $B(r) = (\mu e/2\pi)K_0(\mu r)$. K_0 is the first-order modified Bessel function of the third kind [9]‡. The multiplicative constants in these solutions are adjusted so that in the limit $\mu \to 0$, the usual electrostatic point charge field in $(2+1)$ dimensions is recovered. The magnetic flux produced by a vortex is $\Phi_v = e/\mu$. Then the magnetic potential is of the type described by (4), $A_\theta(r) = (e/2\pi\mu)\,\mathrm{d}K_0(\mu r)/\mathrm{d}r$. Assume that the vortex charge distribution, $\rho(\boldsymbol{r}) = e\sum_k \delta^2(\boldsymbol{r} - \boldsymbol{r}_k(t))$, is in a moving frame where the presence of any currents can be disregarded ($j_i(\boldsymbol{r}) = 0$). Thus the only dynamics left is contained in the fermions whose Schrödinger equation is given by a MVQH with specific potentials $\boldsymbol{A}$ and V. Equation (8) becomes, in this situation,

$$\mu = (1/N)e^2/hc. \tag{9}$$

5. Conclusion

To be able to claim that a fractional Hall conductance follows from a QMVH, one must interpret this theory in a suitable way. This means the inclusion of the following assumptions.

(i) The QMVH describes the carriers in the frame where the transverse external electric field is absent.

(ii) The total intrinsic vortex magnetic flux is in fact the external magnetic flux that strikes the plane ($\Phi_{\mathrm{int}} = \Phi_{\mathrm{ext}}$).

(iii) Vortices have a fixed radius. Consequently they exist whenever the *external* magnetic flux satisfies the vortex flux quantisation condition.

(iv) Vortices are in a close packing arrangement.

The upshot of this interpretation of the quantum Hamiltonian is that, granted the conditions above, the vortex picture leads to (1). Conditions (ii) and (iii) together with (8) do so, the density n being the number of vortices divided by the system's total area. Then a boost back to the laboratory frame will provide the conductivity values of (2) for a filling factor $\nu = 1/N$. For the phenomenological vortex previously defined the two parameters are the magnetic field B_0 and the vortex radius r_0. The radius is constant and so the vortices do not exist whenever the external magnetic field surpasses the discrete values given by the flux quantisation ($B_0\pi r_0^2 = Nhc/e$).

† Some authors [8] have shown that in $(2+1)$ dimensions, a Dirac electron in the presence of an external magnetic field induces the anomalous contribution μ in the action by radiative corrections.

‡ The modified Bessel function $K_0(a)$ has the following asymptotic limits: $a \ll 1$, $K_0(a) \to -\ln(a/2) + 0.5772\ldots$, $a \gg 1$, $K_0(a) \to [(\pi/2a)^{1/2}]\exp(-a)$.

Now I would like to comment on a possible application of this present QMVH approach to the fractional Hall effect observed in the GaAs-AlGaAs heterojunctions [10]. Following the interpretation of a vortex in terms of an electron and the external magnetic field, the main obstacle to this possible connection is pointed out.

Consider the moving frame where the only external field at the plateaux† is magnetic. As it is well known the resistivity in the longitudinal direction drops to zero in the plateaux. The transverse resistivity and conductivity become the inverse of each other. Take the plane of the two-dimensional electron gas. The electrons' total magnetic flux crossing the plane is zero. Therefore the total magnetic flux in the plane is the same external magnetic flux hitting this plane. Then the intrinsic magnetic flux Φ_v is interpreted as the external magnetic flux penetrating the region where an electron loop current exists. The vortex magnetic field $B(r)$ represents the total magnetic field in the same region, i.e. the sum of the external magnetic field striking the plane in these surroundings plus the internal magnetic field produced by the electron's orbital motion. In a first approximation one expects the latter to be negligible when compared to the former. To another particle living in this plane, this electron loop current is seen as a vortex, i.e. an actual source of magnetic field. As seen in the quantum Hamiltonian (3), the dynamics is solely created by the magnetic and electric fields of the vortices. This is as if the external magnetic field pierced the plane producing a vortex at each area of penetration. The QMVH is phenomenological at best because it does not and cannot provide the properties of the loop current produced by one electron. The vortex should be seen as an idealisation of the present description.

Vortices must be close packed to produce a total magnetic field as uniform as possible throughout the plane. The vortex density is high so that the total area divided by their number is S, the area where most of the magnetic flux setting a single vortex is concentrated.

It is unavoidable in the present QMVH formulation that in the charged plane the sum of the individual vortex magnetic fields will not lead to a constant total magnetic field. Even under a tight-packing arrangement there will be local variations of the order of ten to twenty per cent from the minimum to the maximum value. For the electrons trapped into an AlGa-AlGaAs heterostructure one does not expect such a strong diamagnetism. However, the QMVH approach does not claim to describe the magnetic fields infinitesimally away from the vortex two-dimensional plane. Thus in principle one can simply hypothesise a magnetic field approaching uniformity very quickly below and above this plane, so that current experiments would not detect such non-uniformity. The reasons for this possible behaviour are not clear however.

The spatial configuration that vortices take in the plane will be the one that minimises the energy of the QMVH. The question regarding the existence of a gap in the QMVH is of importance to the present approach. This dynamical question will be the subject of a forthcoming publication.

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† For a review on the FQHE see [10].

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